Sensitivity analysis for optimization under constraints and with uncertainties

Sensitivity analysis on (excursion) sets

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System & black-box model

Input parameters :
-component size
-material
-swell height
-...

inputs →

Complex system : Floating wind turbine

output →

-Cost
-energy production
-environmental impact
-...

System & black-box model

- The x are the deterministic inputs
- The u are uncertain inputs : $u = U(\omega)$ with U a random vector of density ρ_U
- f is the objective function to minimize
- g is the constraint function defining the constraint to respect : $g \leq 0$

Optimization problem

Robust optimization problem

$$x^* = \arg\min_x \mathbb{E}[f(x, U)] \quad \text{s.t. } \mathbb{P}[g(x, U) \leq 0] \geq P_{target} \tag{1}$$

Deterministic strategy

$$x^* = \arg\min_x F(x) \quad \text{s.t. } G(x) \leq 0 \tag{2}$$

- Goal Oriented Sensitivity Analysis to reduce the input dimension : Spagnol 2020
- Huge evaluation cost of F and G
- What about the U ?

How to quantify the impact of the uncertain inputs U on the optimization ?

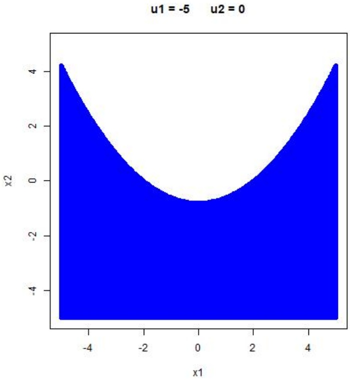
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_2 fixé



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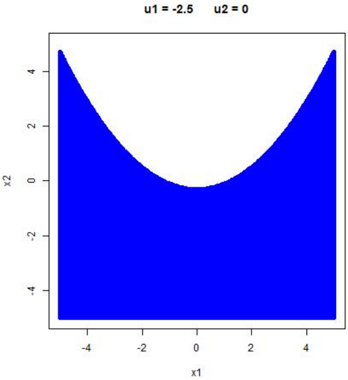

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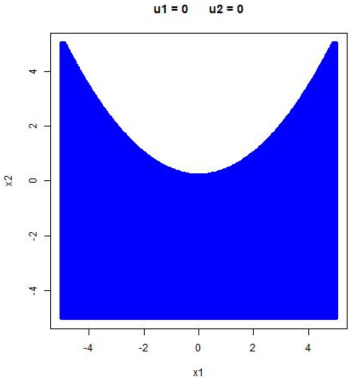

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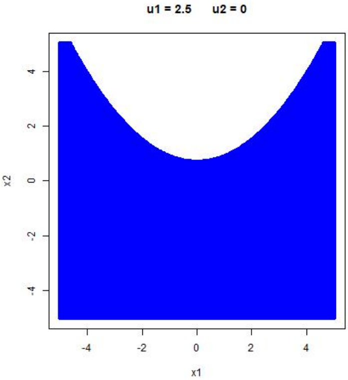

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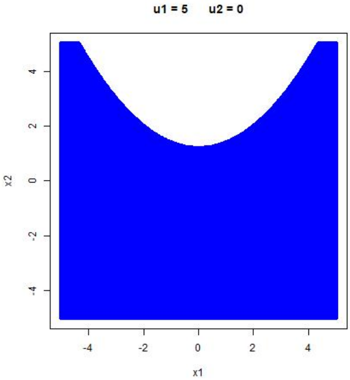

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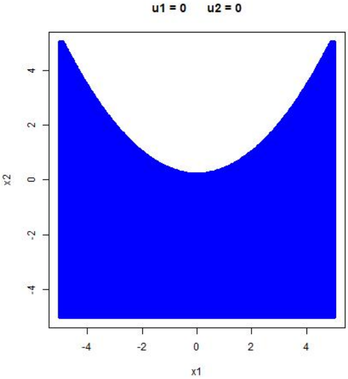
u_1 fixé



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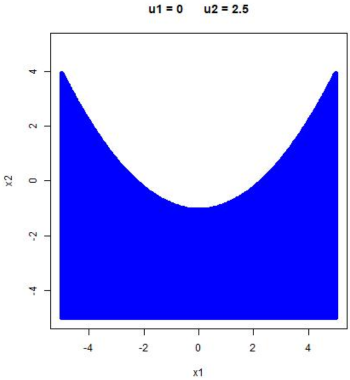

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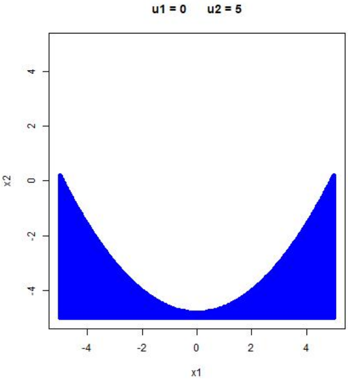

Subproblem : Excursion sets

Excursion sets

New output :

$$\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leq 0\}, \tag{3}$$

which is called a random excursion set.

Influence of the uncertain inputs U on Γ_U ? $\Rightarrow$ SA on excursion sets.
Yet most SA techniques have scalar or vectorial outputs.
How then can we do sensitivity analysis on (excursion) sets ?

- SA on sets using random set theory notably Vorob'ev expectation and deviation
- SA on sets using universal indices from Gamboa et al. 2021
- **SA on sets using RKHS theory**

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Sobol indices on random sets ?

- How to quantify the influence of the input U_i on the output $\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leq 0\}$ which is a random set.
- "Sobol indices" on random sets :

$$S_i = \frac{\text{Var } \mathbb{E}[\Gamma | U_i]}{\text{Var } \Gamma}$$

$\rightarrow$ Expectation and Variance of a random set ?

Vorob'ev expectation and deviation (Molchanov 2005)

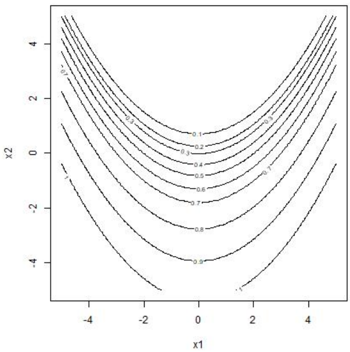
Vorob'ev expectation

$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*} \text{ with } \mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*}) \quad (4)$$

Vorob'ev deviation

$$\mathrm{Var}^V(\Gamma) = \mathbb{E}[\mu(\Gamma \Delta \mathbb{E}^V[\Gamma])], \quad (5)$$

Couverture function, mean volume = 0.545



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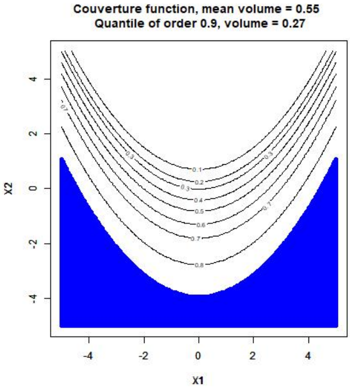

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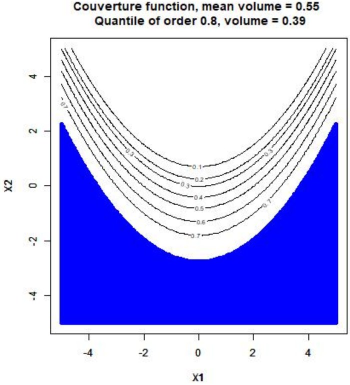

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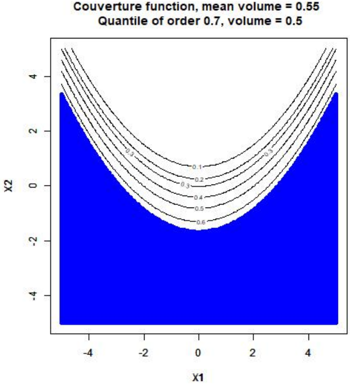

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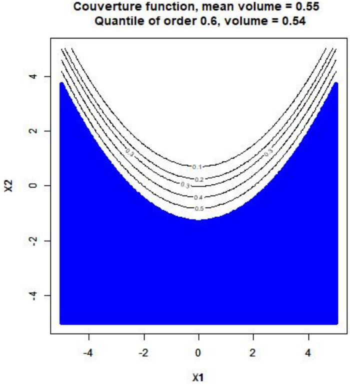

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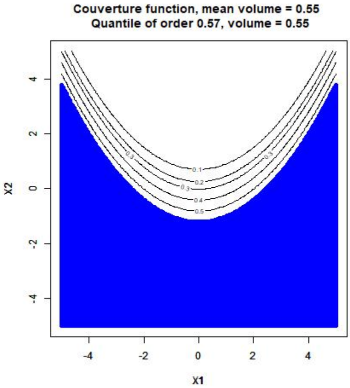

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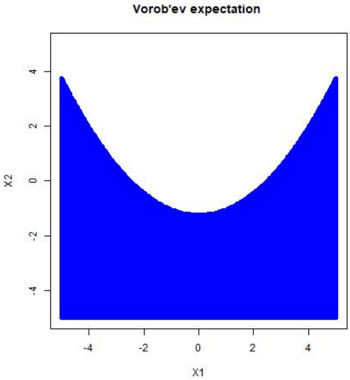


Definition of the proposed indices & estimation

First order Vorob'ev index :

$$S_i^V = \frac{Var^V(\mathbb{E}^V(\Gamma|U_i))}{Var^V(\Gamma)} = \frac{\mathbb{E}[\mu(\mathbb{E}^V(\Gamma|U_i)\Delta\mathbb{E}^V[\mathbb{E}^V(\Gamma|U_i)])]}{\mathbb{E}[\mu(\Gamma\Delta\mathbb{E}^V(\Gamma))]}.$$

But :

$$\mathbb{E}^V[\mathbb{E}^V(\Gamma|U_i)] \neq \mathbb{E}^V(\Gamma)$$

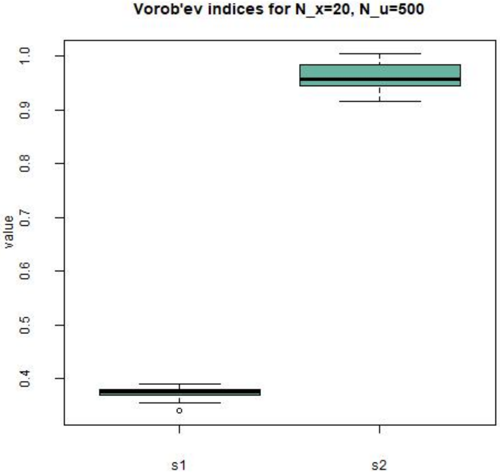
which has two mains issues :

- Very costly Monte Carlo estimation
- No variance decomposition

Results

Toy function

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$



Adaptation of the universal index on random sets

$$S^i_{2,\ \text{Univ}}\ (T_a, \mathbb{Q}) := \frac{\int_{\mathcal{A}} \text{Var}(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]) d\mathbb{Q}(a)}{\int_{\mathcal{A}} \text{Var}\left(T_a(\Gamma_U)\right) d\mathbb{Q}(a)}$$

We use $T_a(\Gamma)$ defined by :

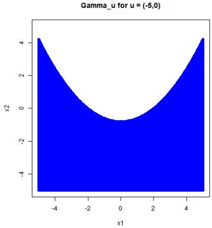
$$T_a(\Gamma) = \mu(\gamma_a \Delta \Gamma), \tag{6}$$

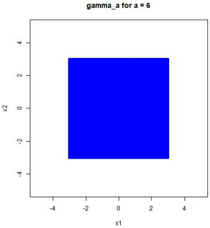
with :

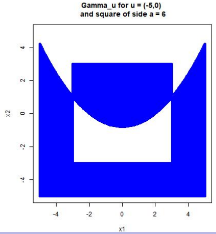
- the symmetric difference Δ defined by $A\Delta B = A \cup B - A \cap B$.
- The volume μ defined by $\mu(\Gamma) = \int_{\mathcal{X}} \mathbb{1}_{x\in\Gamma} dx$.
- $\mathbb{Q}$ is taken uniform on $\mathcal{A}$.
- the γ_a, called test sets, defined through the scalar (or real valued vector) $a \in \mathbb{R}^m$: For instance concentric disks of radius a or concentric squares of side a.

Idea of the universal index

$$S^i_{2,\ \text{Univ}}\ (T_a, \mathbb{Q}) := \frac{\int_{\mathcal{A}} \text{Var}(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]) d\mathbb{Q}(a)}{\int_{\mathcal{A}} \text{Var}\left(T_a(\Gamma_U)\right) d\mathbb{Q}(a)}$$







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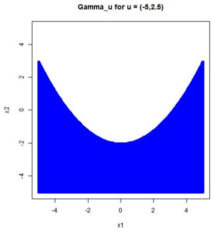

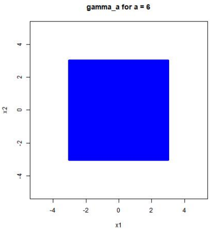

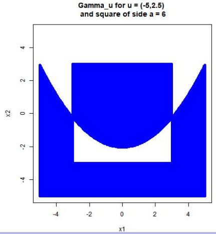

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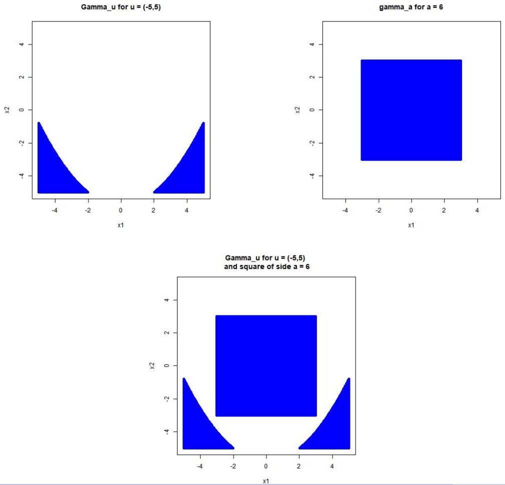

Conclusion on universal indices on sets

Issues

- Choice of T (the symmetric difference)
- Choice of $\mathbb{Q}$
- Choice of the test sets : we used squares and circles.

We could compare realizations to each other $\rightarrow$ Kernel-based sensitivity analysis.

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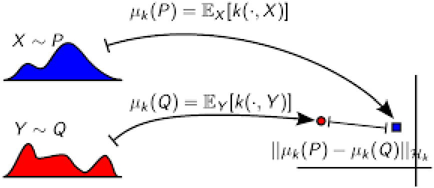
Distribution embedding into a RKHS



Distance between distributions as expectation of kernels, Gretton et al. 2006

$$\gamma_k^2(\mathbb{P}, \mathbb{Q})) := ||\mu_\mathbb{P} - \mu_\mathbb{Q}||^2_{\mathcal{H}_k} = \mathbb{E}_{\substack{X,X' \sim \mathbb{P} \\ Y,Y' \sim \mathbb{Q}}}[k(X, X') + k(Y, Y') - 2k(X, Y)]$$

$\rightarrow$ is a distance between distribution iif k is *characteristic* (i.e. injectivity of the embedding)

MMD-based index

MMD-based index

$$S_i := \mathbb{E}_{X_i}[\gamma_k^2(\mathbb{P}_Y, \mathbb{P}_{Y|X_i})]$$
$$= \mathbb{E}_{X_i}\mathbb{E}_{\xi\sim\mathbb{P}_{Y|X_i}}\mathbb{E}_{\xi'\sim\mathbb{P}_{Y|X_i}}[k(\xi,\xi')] - \mathbb{E}_{Y\sim\mathbb{P}_Y}\mathbb{E}_{Y'\sim\mathbb{P}_Y}[k(Y,Y')].$$

$\rightarrow$ has a ANOVA decomposition (daVeiga 2021) : can be used to rank the inputs by influence.

Estimation

- Pick & freeze estimation
- Rank-based estimation

HSIC-based index Gretton et al. 2006

HSIC-based index

$$\begin{aligned} HSIC_k(X_i, Y) :&= \gamma_k^2(\mathbb{P}_{X_i,Y}, \mathbb{P}_{X_i} \otimes \mathbb{P}_Y) \\ &= \mathbb{E}_{X_i,X_i',Y,Y'} k_{\mathcal{X}}(X_i, X_i') k(Y, Y') \\ &+ \mathbb{E}_{X_i,X_i'} k_{\mathcal{X}}(X_i, X_i') \mathbb{E}_{Y,Y'} k(Y, Y') \\ &- 2\mathbb{E}_{X_i,Y}[\mathbb{E}_{X_i'} k_{\mathcal{X}}(X_i, X_i') \mathbb{E}_{Y'} k(Y, Y')] \end{aligned}$$

with (X_i', Y') an independent copy of (X_i, Y).

$HSIC_k(X_i, Y) = 0$ iif $X_i \perp Y$ (when k characteristic)$\rightarrow$ suited to identify the negligible inputs through independence testing.

Estimation

- Biased or unbiased classic estimators of $HSIC_k(X_i, Y)$
- p-values computable through asymptotic results or bootstrap methods

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SA on sets : a kernel between sets

Proposition (A kernel between sets, Balança et Herbin 2012)

The function $k_{set} : \mathcal{F}(\mathcal{X}) \times \mathcal{F}(\mathcal{X}) \to \mathbb{R}$ *defined by*

$$\forall \Gamma, \Gamma' \in \mathcal{F}(\mathcal{X}), \quad k_{set}(\Gamma, \Gamma') = \frac{\sigma^2}{2\lambda} e^{-\lambda \mu(\Gamma \Delta \Gamma')},$$

is positive definite for any positive scalars σ *and* λ*.*

Moore-Aronszajn theorem (Aronszajn (1950)) gives then the existence of a unique RKHS $\mathcal{H} \subset \mathcal{F}(\mathcal{X})^{\mathbb{R}}$ of reproducing kernel k_{set}.

Now we need to embed "random sets distributions" in $\mathcal{H}$

Random set distribution embedding

Definition (Capacity functional)

The capacity functional of a random closed set Γ *denoted* T_Γ *is defined by :*

$$T_\Gamma : \begin{array}{ccc} \mathcal{K}(\mathcal{X}) & \rightarrow & [0,1] \\ K & \mapsto & \mathbb{P}(\Gamma \cap K \neq \emptyset). \end{array} \quad (7)$$

Definition (Mean embedding of a capacity functional)

The mean embedding of T_Γ *is defined as*

$$\mu_\Gamma = \mathbb{E}[k_{set}(\Gamma, \cdot)]. \quad (8)$$

Estimation with set-valued outputs

In the MMD- and HSIC-based indices expressions, we need to estimate quantities of the form $\mathbb{E}k_{set}(\Gamma_1, \Gamma_2)$.
With sets, $k_{set}(\Gamma_1, \Gamma_2)$ also require an estimation.

$$
\begin{aligned}
k_{set}(\Gamma_1, \Gamma_2) &= e^{-\mu(\Gamma_1 \Delta \Gamma_2)}] && (9)\\
&= e^{-\mathbb{E}[\mathbb{1}_{\Gamma\Delta\Gamma'}(X)]} \text{ with } X \sim \mathcal{U}(\mathcal{X}) && (10)\\
&\simeq e^{-\mu(\mathcal{X})\frac{1}{N_X}\sum_{i=1}^{N_X}\mathbb{1}_{\Gamma_1\Delta\Gamma_2}(X^i)} && (11)\\
&= \widehat{k_{set}}(\Gamma_1, \Gamma_2) && (12)
\end{aligned}
$$

Then we inject it in the indices estimators. For instance the normalized MMD-based index estimated through pick and freeze method is :

$$
\widehat{S_{A,p.f}^{MMD}} = \frac{\sum_{i=1}^{n} \widehat{k_{set}}(\Gamma^{(i)}, \tilde{\Gamma}^{A,(i)}) - \widehat{k_{set}}(\Gamma^{(i)}, \Gamma'^{(i)})}{\sum_{i=1}^{n} \widehat{k_{set}}(\Gamma^{(i)}, \Gamma^{(i)}) - \widehat{k_{set}}(\Gamma^{(i)}, \Gamma'^{(i)})}. \quad (13)
$$

Asymptotic behaviour of the indices on sets

Proposition (Quadratic error of a nested Monte Carlo estimator)

With the previous notations, using Rainforth *et al. 2018, we have*

$$\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}\widehat{k_{set}}(\Gamma_1^{(j)},\Gamma_2^{(j)}) - \mathbb{E}k_{set}(\Gamma_1,\Gamma_2)\right)^2 = \mathcal{O}(\frac{1}{n}+\frac{1}{N_x^2}). \quad (14)$$

With this result, we can show that each quadratic error of our indices on sets has the same asymptotic behavior with rate $\mathcal{O}(\frac{1}{n}+\frac{1}{N_x^2})$

Toy function, El-Amri et al. 2021

$$\forall x, u \in [-5,5]^2 \times [-5,5]^2 \quad g(x,u) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1. \tag{15}$$

$$\Gamma_U = \{x \in \mathcal{X}, g(x,U) \leq 0\}, \tag{16}$$

Toy function	Index	U_1	U_2
$g = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$	$\widehat{S_{i,p.f}^{MMD}}$	0.074	0.458
	$\widehat{S_{i,p.f}^{T,MMD}}$	0.545	0.935

Table – MMD-based indices

Toy function	Index	U_1	U_2
$g = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$	$\widehat{S_i^{HSIC_u}}$	0.072	0.284
	p-value	$4.70e-03$	$5.97e-10$

Table – HSIC-based indices

Test case : oscillator, Cousin 2021

$$g_1(\boldsymbol{x}, \boldsymbol{u}) = u_{r_1} - \max_{t \in [0,T]} \mathcal{Y}'(x_1 + u_1, x_2 + u_2, u_p; t), \tag{17}$$

$$g_2(\boldsymbol{x}, \boldsymbol{u}) = u_{r_2} - \max_{t \in [0,T]} \mathcal{Y}''(x_1 + u_1, x_2 + u_2, u_p; t), \tag{18}$$

with $\mathcal{Y}$ the solution of the harmonic oscillator defined by :

$$(x_1 + u_1)\mathcal{Y}''(t) + u_p\mathcal{Y}'(t) + (x_2 + u_2)\mathcal{Y}(t) = \eta(t). \tag{19}$$

Uncertainty	Distribution	Uncertainty	Distribution
U_1	$\mathcal{U}\left[-0.3, 0.3\right]$	U_{r_1}	$\mathcal{N}\left(1, 0.1^2\right)$
U_2	$\mathcal{U}\left[-1, 1\right]$	U_{r_2}	$\mathcal{N}\left(2.5, 0.25^2\right)$
U_p	$\mathcal{U}[0.5, 1.5]$	U_{r_3}	$\mathcal{N}\left(15, 3^2\right)$

Table – Definition of the uncertain inputs

Results on the oscillator case : MMD-based index

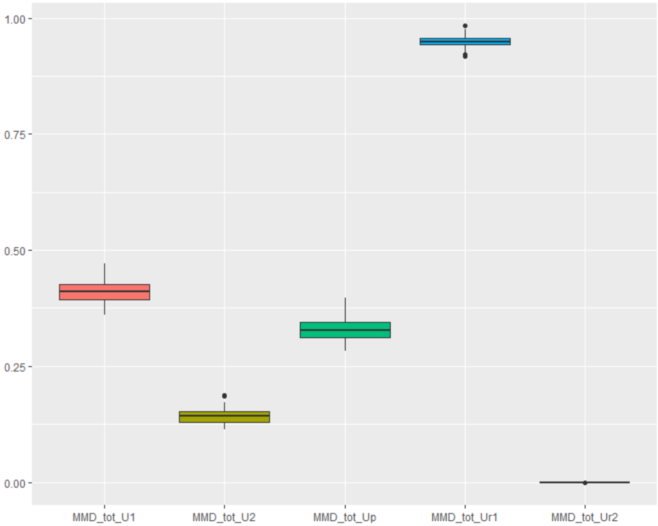

Figure – MMD-based total index for the constraint $g_1 \leq 0$

Results on the oscillator case : MMD-based index

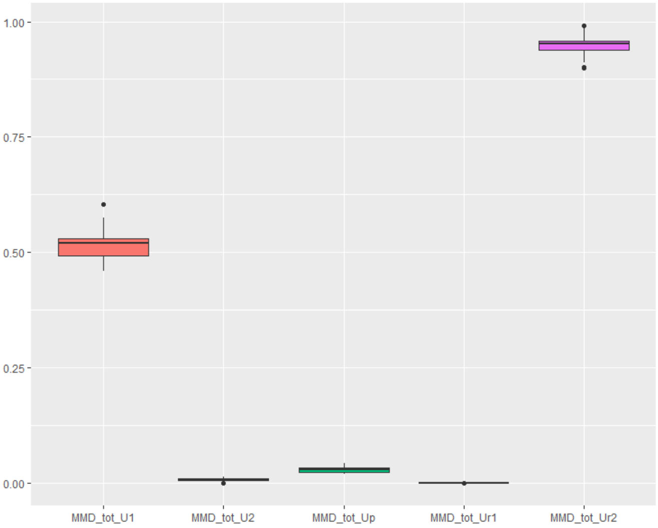

Figure – MMD-based total index for the constraint $g_2 \leq 0$

Results on the oscillator case : MMD-based index

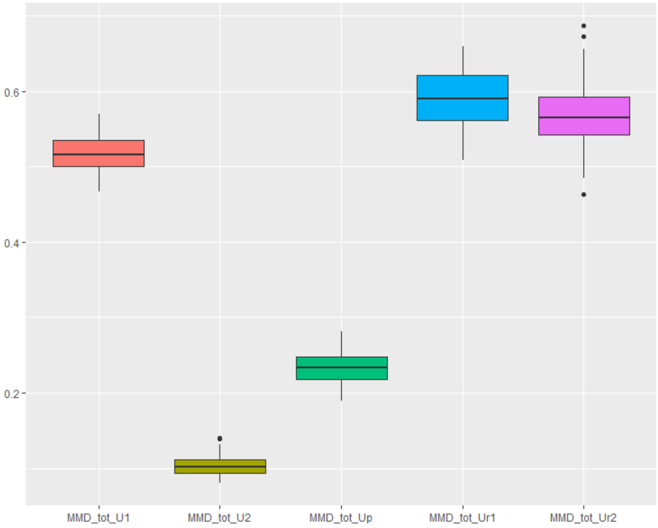

Figure – MMD-based total index for the constraint $g1 \leq 0$ and $g_2 \leq 0$

Results on the oscillator case : MMD-based index

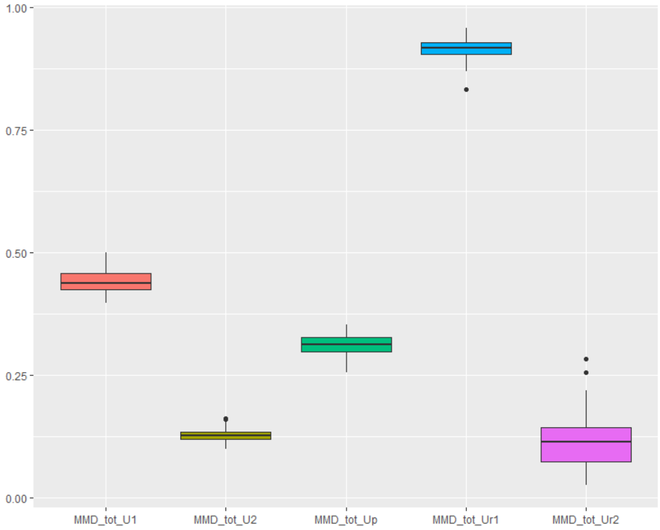

Figure – MMD-based total index for the couple ($g_1 \leq 0, g_2 \leq 0$)

Results on the oscillator case : HSIC-based index

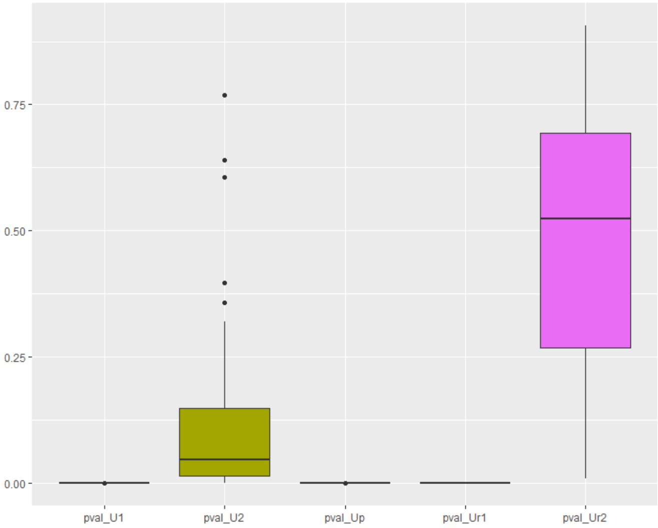

Figure – p-value of the HSIC-based index for the constraint $g_1 \leq 0$

Results on the oscillator case : HSIC-based index

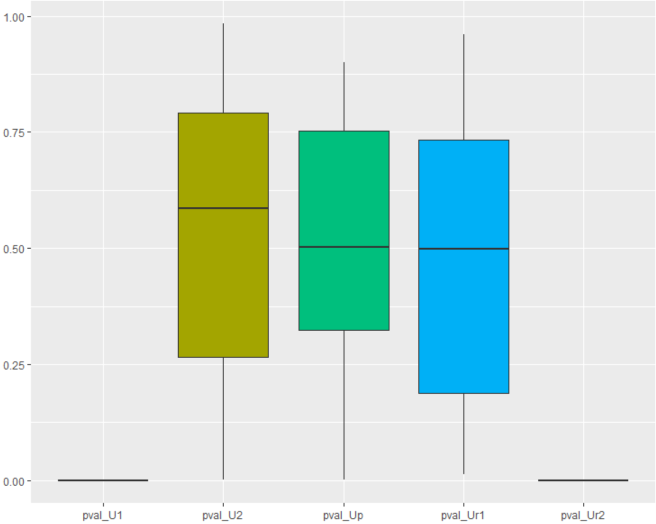

Figure – p-value of the HSIC-based index for the constraint $g_2 \leq 0$

Results on the oscillator case : HSIC-based index

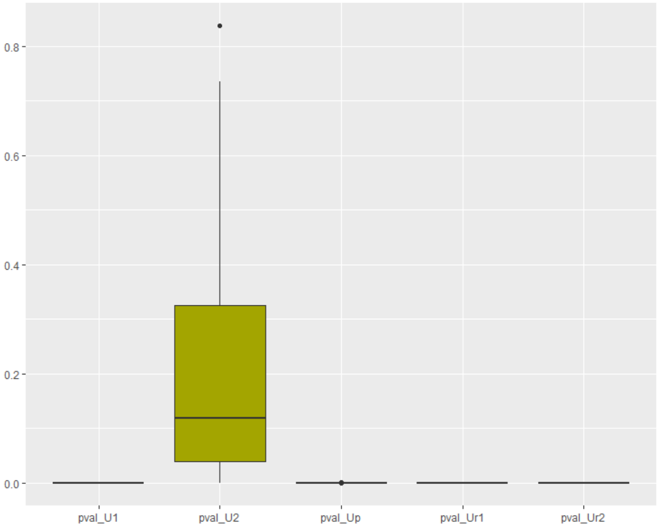

Figure – p-value of the HSIC-based index for the constraint $g1 \leq 0$ and $g_2 \leq 0$

Results on the oscillator case : HSIC-based index

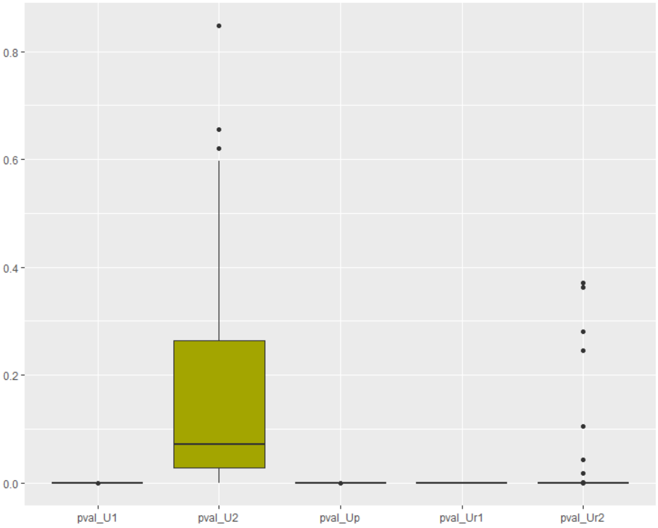

Figure – p-value of the HSIC-based index for the couple $(g_1 \leq 0, g_2 \leq 0)$

Conclusion

Kernel-based SA on set-valued outputs (paper "soon")

- A way to do SA on set-valued outputs
- On excursion sets : An answer to "How to do SA on the uncertain inputs in the context of robust optimization ?"

Future work

- Test the three methods on a real test case (of Adan Reyes Reyes from IFPEN)
- Use it inside an optimization

El-Amri, Reda et al. (2021). *A sampling criterion for constrained Bayesian optimization with uncertainties*. arXiv : 2103.05706 [stat.ML].

Aronszajn, N. (1950). "Theory of reproducing kernels". In : *Transactions of the American Mathematical Society* 68.3, p. 337-404. issn : 0002-9947, 1088-6850. doi : 10.1090/S0002-9947-1950-0051437-7. url : https://www.ams.org/tran/1950-068-03/S0002-9947-1950-0051437-7/ (visité le 02/11/2022).

Balança, Paul et Erick Herbin (jan. 2012). "A set-indexed Ornstein-Uhlenbeck process". In : *Electronic Communications in Probability* 17.none. doi : 10.1214/ecp.v17-1903. url : https://doi.org/10.1214%2Fecp.v17-1903.

Cousin, Alexis (oct. 2021). "Chance constraint optimization of a complex system : Application to the design of a floating offshore wind turbine". Theses. Institut Polytechnique de Paris. url : https://tel.archives-ouvertes.fr/tel-03500604.

daVeiga, Sébastien (jan. 2021). "Kernel-based ANOVA decomposition and Shapley effects - Application to global sensitivity analysis". working paper or preprint. url : https://hal.archives-ouvertes.fr/hal-03108628.

Gamboa, Fabrice et al. (août 2021). "Sensitivity analysis in general metric spaces". In : *Reliability Engineering and System Safety*. doi : 10.1016/j.ress.2021.107611. url : https://hal.archives-ouvertes.fr/hal-02044223.

Gretton, Arthur et al. (2006). “A Kernel Method for the Two-Sample-Problem”. In : *Advances in Neural Information Processing Systems*. T. 19. MIT Press. url : https://proceedings.neurips.cc/paper/2006/hash/e9fb2eda3d9c55a0d89c98d6c54b5b3e-Abstract.html.

Molchanov, Ilya (2005). *Theory of random sets*. T. 19. 2. Springer.

Rainforth, Tom et al. (23 mai 2018). *On Nesting Monte Carlo Estimators*. arXiv : 1709.06181[stat]. url : http://arxiv.org/abs/1709.06181.

Spagnol, Adrien (juill. 2020). “Kernel-based sensitivity indices for high-dimensional optimization problems”. Theses. Université de Lyon. url : https://tel.archives-ouvertes.fr/tel-03173192.

Excursion sets are random sets

For K compact, $h(U) = X = \{x \in \mathcal{X}, g(x, U) \leq 0\}$

$$\begin{aligned}\{K \cap X \neq \emptyset\} &=^c \{\omega, K \cap X(\omega) = \emptyset\} \\ &=^c \{\omega, \forall x \in K \ g(x, U(\omega)) > 0\} \\ &=^c \{\omega, \inf_{x \in K} g(x, U(\omega)) > 0\} \text{ as } K \text{ compact and g continuous in } x \\ &=^c U^{-1}(\inf_{x \in K} g(x, \cdot)^{-1}(]0, +\infty[)) \in \mathcal{F}\end{aligned}$$

Indices estimation

$$S_{A,p.f}^{MMD} = \frac{\sum_{i=1}^{n} k_{set}(\Gamma^{(i)}, \tilde{\Gamma}^{A,(i)}) - k_{set}(\Gamma^{(i)}, \Gamma'^{(i)})}{\sum_{i=1}^{n} k_{set}(\Gamma^{(i)}, \Gamma^{(i)}) - k_{set}(\Gamma^{(i)}, \Gamma'^{(i)})}. \quad (20)$$

$$S_{l,rank}^{MMD} = \frac{\frac{1}{n}\sum_{i=1}^{n} k_{set}(\Gamma^{(i)}, \Gamma^{\sigma_n^l(i)}) - \frac{1}{n^2}\sum_{i,j=1}^{n} k_{set}(\Gamma^{(i)}, \Gamma^{(j)})}{\frac{1}{n}\sum_{i=1}^{n} k_{set}(\Gamma^{(i)}, \Gamma^{(i)}) - \frac{1}{n^2}\sum_{i,j=1}^{n} k_{set}(\Gamma^{(i)}, \Gamma^{(j)})}. \quad (21)$$

$$\mathrm{HSIC}_u(U_A, \Gamma) = \frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} \left(k_{\mathcal{U}_A}\left(U_A^{(i)}, U_A^{(j)}\right) - 1\right) k_{set}\left(\Gamma^{(i)}, \Gamma^{(j)}\right), \quad (22)$$

$$\mathrm{HSIC}_b(U_A, \Gamma) = \frac{1}{n^2} \sum_{i,j=1}^{n} \left(k_{\mathcal{U}_A}\left(U_A^{(i)}, U_A^{(j)}\right) - 1\right) k_{set}\left(\Gamma^{(i)}, \Gamma^{(j)}\right). \quad (23)$$