Sensitivity analysis for optimization under constraints and with uncertainties

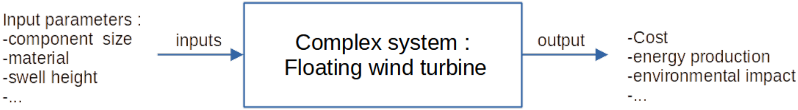
Sensitivity analysis on excursion sets

Noé Fellmann
Céline Helbert & Christophette Blanchet (ECL)
Adrien Spagnol & Delphine Sinoquet (IFPEN)

École Centrale de Lyon & IFP Énergie nouvelles

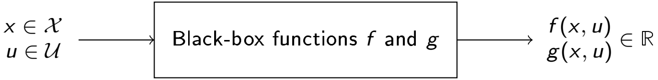
Journées CIROQUO 23, 24 et 25 mai 2022

System


Input parameters :
-component size
-material
-swell height
-...
inputs
Complex system :
Floating wind turbine
output
-Cost
-energy production
-environmental impact
-...


Source: dnv

Black-box model & optimization problem



- The x are the deterministic inputs
- The u are uncertain inputs : $u = U(\omega)$ with U a random vector of density ρ_U
- f is the objective function to minimize
- g is the constraint function defining the constraint to respect : $g \leq 0$

Optimization problem

$$\begin{aligned} x^* &= \arg\min \mathbb{E}_U[f(x,U)] \\ &\text{s.t. } \mathbb{P}_U[g(x,U) \leq 0] \geq P_{target} \end{aligned} \tag{1}$$

Sensitivity analysis adapted to optimization

Sensitivity analysis (SA) = Studying the impact of each input on the variability of the output.
$\Rightarrow$ SA + Optimization : Studying the impact of each input on the minimization of the objective function and on the respect of the constraints.

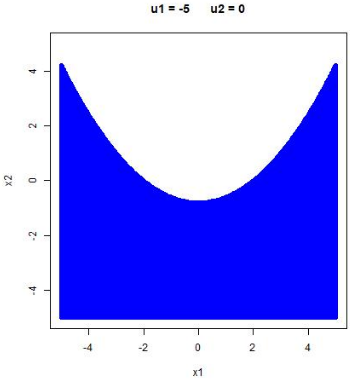
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_2 fixé



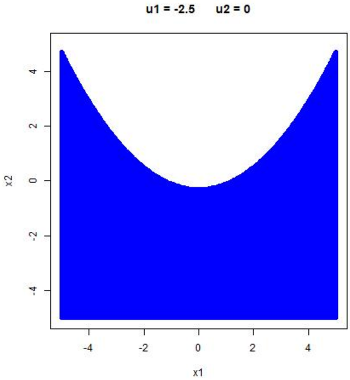
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x, u) \in [-5, 5]^4 \;\; g(\boldsymbol{x}, \boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5, 5]^2, g(x, u) \leq 0\}$$

u_2 fixé



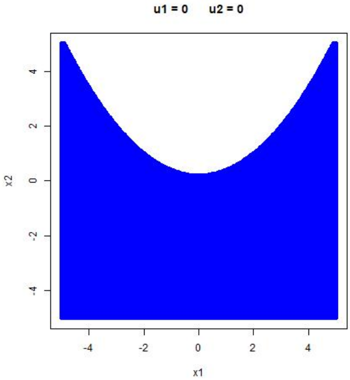
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_2 fixé



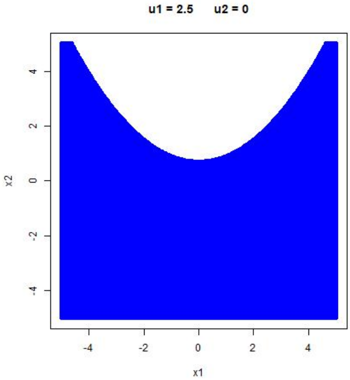
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_2 fixé



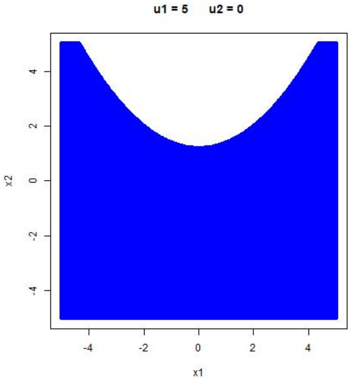
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_2 fixé



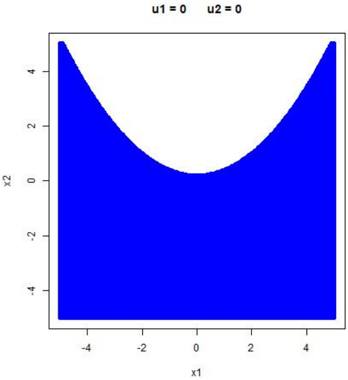
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x, u) \in [-5, 5]^4 \;\; g(\boldsymbol{x}, \boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5, 5]^2, g(x, u) \leq 0\}$$

u_1 fixé



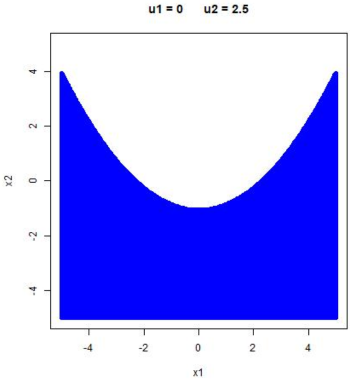
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5,5]^2, g(x,u) \leq 0\}$$

u_1 fixé



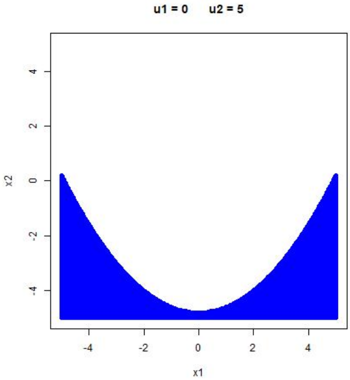
Toy function

Toy function from [El-Amri et al. 2021]

$$\forall (x, u) \in [-5, 5]^4 \;\; g(\boldsymbol{x}, \boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_u = \{x \in [-5, 5]^2, g(x, u) \leq 0\}$$

u_1 fixé



Subproblem : Excursion sets

Excursion sets

New output :

$$\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leq 0\}, \tag{2}$$

which is called a random excursion set.

Influence of the uncertain inputs U on Γ_U ? $\Rightarrow$ SA on excursion sets.
Yet most SA techniques have scalar or vectorial outputs.
How then can we do sensitivity analysis on (excursion) sets ?

Table of Contents

Table of Contents

Random set theory [Molchanov 2005]

The natural definition of a random set is as a set-valued random variable X i.e. as a measurable function :

$$X : \begin{array}{ccc} (\Omega, \mathcal{F}) & \rightarrow & (\mathcal{K}(\mathcal{X}), \mathcal{B}(\mathcal{K}(\mathcal{X}))) \\ \omega & \longmapsto & X(\omega) \end{array}. \tag{3}$$

The notion of measurability depends on the definition of $\mathcal{B}(\mathcal{K}(\mathcal{X}))$ which depends on the topology we choose for $\mathcal{K}(\mathcal{X})$.
From [Molchanov 2005], a notion of measurability can be written as :

$$\forall K \in \mathcal{K}(\mathcal{X}), X^{-} = \{\omega \in \Omega, X(\omega) \cap K \neq \emptyset\} \in \mathcal{F}. \tag{4}$$

With this definition of measurability, we can show that our sets $\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leq 0\}$ are random compact sets (see 29).

Sobol indices on random sets ?

- We want to characterize the influence of the input U_i on the output $\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leq 0\}$ which is a random set.
- Sobol indices on random sets : We would want to compute $S_i = \frac{\text{Var } \mathbb{E}[\Gamma | U_i]}{\text{Var } \Gamma} \rightarrow$ The difficulty lies in defining the (conditional) expectation and the variance of a random set.

Vorob'ev expectation and deviation [Molchanov 2005]

The Vorob'ev expectation of a random set Γ is defined as the Vorob'ev quantile which volume is equal (or closer) to the mean volume of Γ :

$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}, \tag{5}$$

with α^* defined by :

$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*}). \tag{6}$$

Or if the previous equation has no solution, from :

$$\mu(Q_\alpha) \leq \mathbb{E}[\mu(\Gamma)] \leq \mu(Q_{\alpha^*}) \quad \forall \alpha > \alpha^*. \tag{7}$$

Then, the Vorob'ev deviation is defined by :

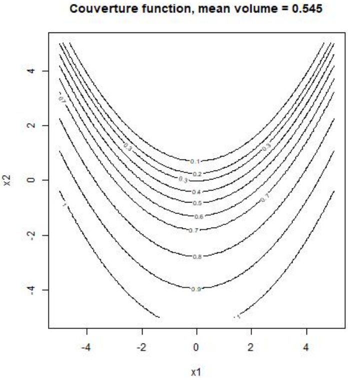
$$\mathrm{Var}^V(\Gamma) = \mathbb{E}[\mu(\Gamma \Delta \mathbb{E}^V[\Gamma])], \tag{8}$$

with the symmetric difference Δ defined by $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Example of Vorob'ev expectation

Vorob'ev expectation

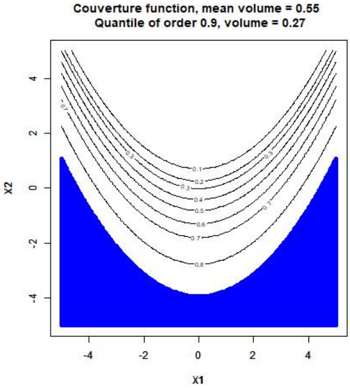
$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$



Example of Vorob'ev expectation

Vorob'ev expectation

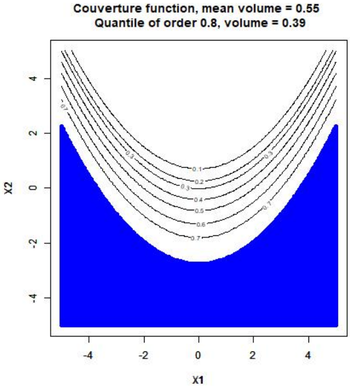
$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$



Example of Vorob'ev expectation

Vorob'ev expectation

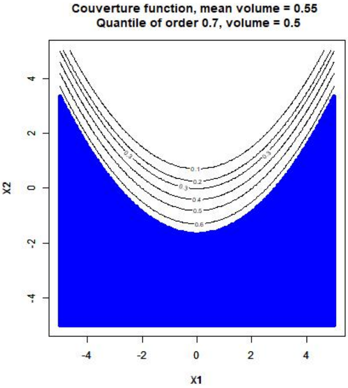
$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$



Example of Vorob'ev expectation

Vorob'ev expectation

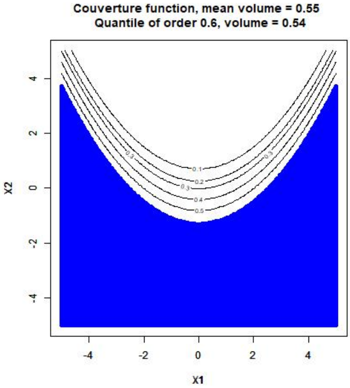
$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$



Example of Vorob'ev expectation

Vorob'ev expectation

$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$

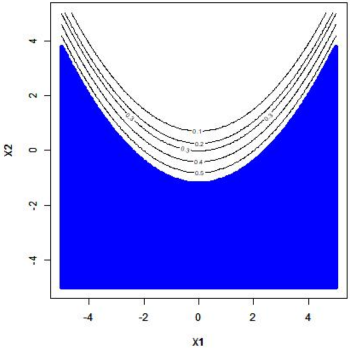


Example of Vorob'ev expectation

Vorob'ev expectation

$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$

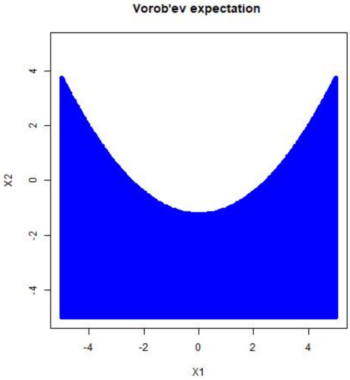
Couverture function, mean volume = 0.55
Quantile of order 0.57, volume = 0.55



Example of Vorob'ev expectation

Vorob'ev expectation

$$\mathbb{E}^V[\Gamma] = \{x \in \mathcal{X}, \mathbb{P}(x \in \Gamma) \geq \alpha^*\} = Q_{\alpha^*}$$
$$\mathbb{E}[\mu(\Gamma)] = \mu(Q_{\alpha^*})$$



Definition of the proposed indices & estimation

Using the previous definitions, we still need a definition of conditional Vorob'ev expectation. We propose the following :

$$\mathbb{E}^V(\Gamma|U_1) = \omega \mapsto \mathbb{E}^V(\Gamma|U_1(\omega)) \quad \text{where} \quad \Gamma|U_1(\omega) = \{x \in \mathcal{X}, g(x, U_1(\omega), U_2) \leq 0\}.$$

With this definition, the first order Vorob'ev index is :

$$S_i^V = \frac{Var^V(\mathbb{E}^V(\Gamma|U_i))}{Var^V(\Gamma)} = \frac{\mathbb{E}[\mu(\mathbb{E}^V(\Gamma|U_i)\Delta\mathbb{E}^V[\mathbb{E}^V(\Gamma|U_i)])]}{\mathbb{E}[\mu(\Gamma\Delta\mathbb{E}^V(\Gamma))]}.$$

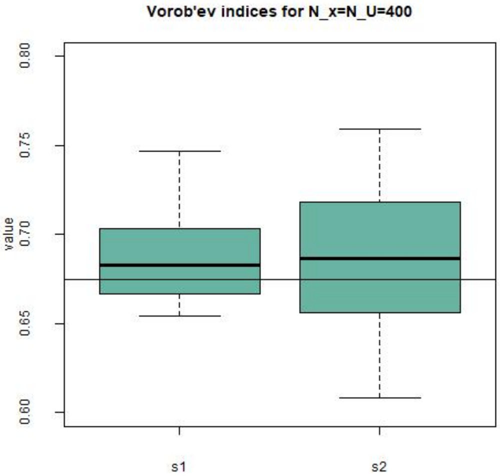
Estimation

- Monte Carlo estimation
- Issue : $\mathbb{E}^V[\mathbb{E}^V(\Gamma|U_i)] \neq \mathbb{E}^V(\Gamma) \rightarrow$ Coslty estimation

Results

Toy function 1

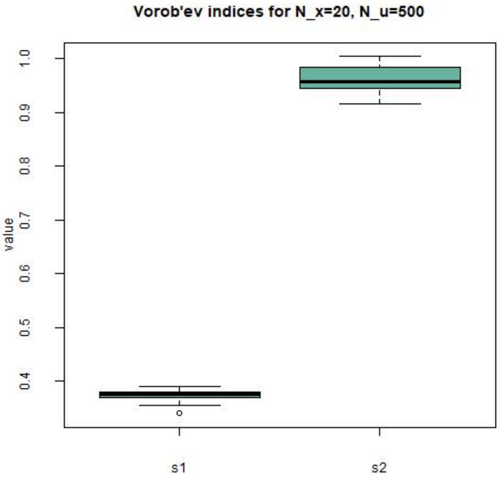
$$\forall (x, u_1, u_2) \in [0, 1]^3 \quad g(x, u) = x - u_1 - u_2$$



Results

Toy function 2

$$\forall(x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$



Results

With $N_x = 20$ and $N_u = 500$

g function	Index	Results
$-x_1^2 + 5x_2 - u_1 + u_2^2 - 1$	s_1	0.366
	s_2	0.949
	$s_1 + s_2$	1.314
$-x_1^2 + 5x_2 + u_1^2 + u_2^2 - 1$	s_1	0.721
	s_2	0.715
	$s_1 + s_2$	1.436
$-x_1^2 + 5x_2 + u_1^2 + u_2^2 + u_1 u_2 - 1$	s1	0.598
	s_2	0.547
	$s_1 + s_2$	1.146
$-x_1^2 + 5x_2 + u_1^2 - 1$	s_1	1.00
	s_2	0.00
	$s_1 + s_2$	1.00
$-x_1^2 + 5x_2 + u_1^2 + u_1 u_2 - 1$	s_1	0.721
	s_2	0.034
	$s_1 + s_2$	0.755

Conclusion on SA through random sets

- The index quantifies an influence of the inputs on the excursion set
- Very costly estimation
- No variance decomposition

Table of Contents

Definition of the universal index

Index introduced in [Gamboa et al. 2021] and [Fort, Klein et Lagnoux 2021] which generalizes many existing indices, and only requires the output space to be a metric space.
With $Z = f(U_1, ..., U_p) \in \mathcal{Z}$, the universal sensitivity index with respect to U_i is defined as

$$S^i_{2,\text{ Univ}}\ (T_\cdot, \mathbb{Q}) := \frac{\int_{\mathcal{A}} \text{Var}\left(\mathbb{E}\left[T_a(Z) \mid U_i\right]\right) d\mathbb{Q}(a)}{\int_{\mathcal{A}} \text{Var}\left(T_a(Z)\right) d\mathbb{Q}(a)}, \tag{9}$$

where (T_a) are tests functions defined by :

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{Z} & \to & \mathbb{R} \\ (a, z) & \mapsto & T_a(z), \end{array} \tag{10}$$

and $\mathcal{A}$ is some measurable space endowed with a probability measure $\mathbb{Q}$.
Generalization of existing indices :

- Sobol index : $S^i = S^i_{2,\text{ Univ}}\ (Id, \mathbb{Q})$ with Id beeing the indicator function : $T_a(z) = z$.
- Cramér-von Mises index : $T_a(z) = \mathbb{1}_{x \leq a}$, $\mathbb{Q} = \mathbb{P}$ with $\mathcal{A} = \mathcal{X}$.

Interpretation of the universal index on random sets

With $Z = \Gamma_U$, the index is :

$$S^i_{2,\text{ Univ}}\,(T_\cdot, \mathbb{Q}) := \frac{\int_{\mathcal{A}} \text{Var}(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]) d\mathbb{Q}(a)}{\int_{\mathcal{A}} \text{Var}\left(T_a(\Gamma_U)\right) d\mathbb{Q}(a)},$$

Interpretation :

- Let $a \in \mathcal{A}$, it defines a random variable $T_a(\Gamma_U)$ which contains a part of information on Γ_U : for instance $T_a(\Gamma_u) = a\mu(\Gamma_U)$.
- $\text{Var}(T_a(\Gamma_U)) = \text{Var}\left(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]\right) + \mathbb{E}\left(\text{Var}\left[T_a(\Gamma_U) \mid U_i\right]\right).$
- $\int_{\mathcal{A}} \text{Var}(T_a(\Gamma_U)) = \int_{\mathcal{A}} \text{Var}\left(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]\right) + \int_{\mathcal{A}} \mathbb{E}\left(\text{Var}\left[T_a(\Gamma_U) \mid U_i\right]\right).$

Influence of T_a and $\mathbb{Q}$

What are the impact of the choices of T_a and $\mathbb{Q}$, how to choose them ?

Adaptation of the universal index on random sets

$$S^i_{2,\text{ Univ}}\ (T_a,\mathbb{Q}) := \frac{\int_{\mathcal{A}} \text{Var}(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]) d\mathbb{Q}(a)}{\int_{\mathcal{A}} \text{Var}\left(T_a(\Gamma_U)\right) d\mathbb{Q}(a)}$$

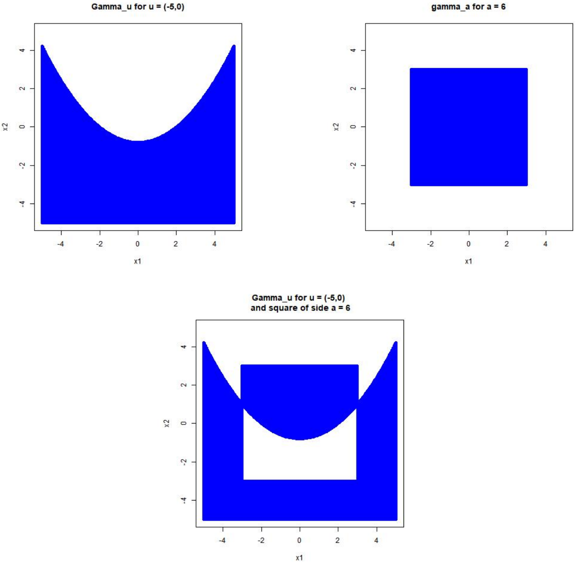
We use $T_a(\Gamma)$ defined by :

$$T_a(\Gamma) = \mu(\gamma_a \Delta \Gamma), \tag{11}$$

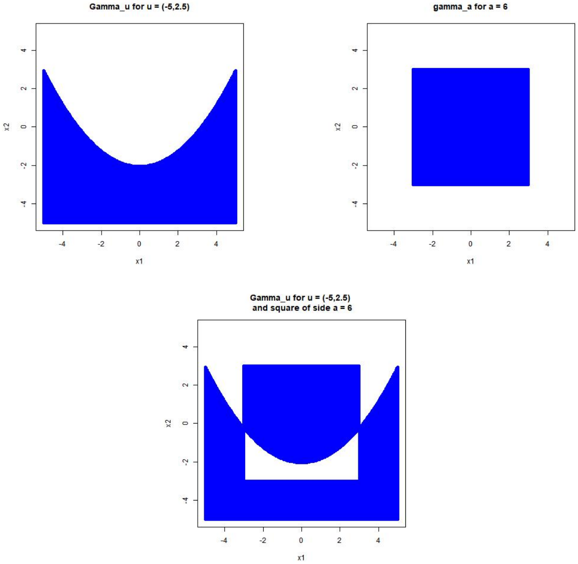
with :

- the symmetric difference Δ defined by $A\Delta B = A \cup B - A \cap B$.
- The volume μ defined by $\mu(\Gamma) = \int_{\mathcal{X}} \mathbb{1}_{x\in\Gamma} dx$.
- $\mathbb{Q}$ is taken uniform on $\mathcal{A}$.
- the γ_a, called test sets, defined through the scalar (or real valued vector) $a \in \mathbb{R}^m$: For instance concentric disks of radius a or concentric squares of side a.

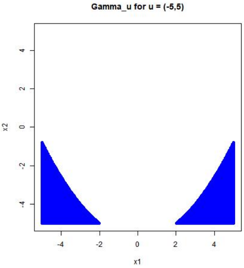
Symmetric difference

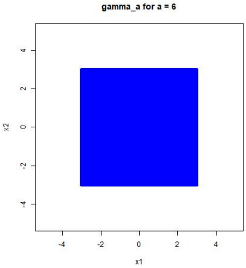


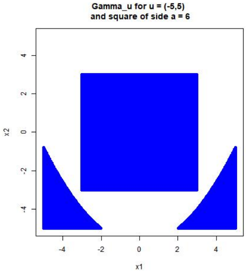
Symmetric difference



Symmetric difference







Estimation

Pick and freeze method : U' independent copy of U,

$$U = \begin{pmatrix} U_1 & \cdots & U_i & \cdots & U_p \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} U_1 & \cdots & U_i' & \cdots & U_p \end{pmatrix}$$

We have then : $\mathrm{Var}(\mathbb{E}\left[T_a(\Gamma_U) \mid U_i\right]) = \mathrm{Cov}(T_a(\Gamma_U), T_a(\Gamma_{\tilde{U}}))$.

Estimation $\mu(\mathcal{X}) = 1$

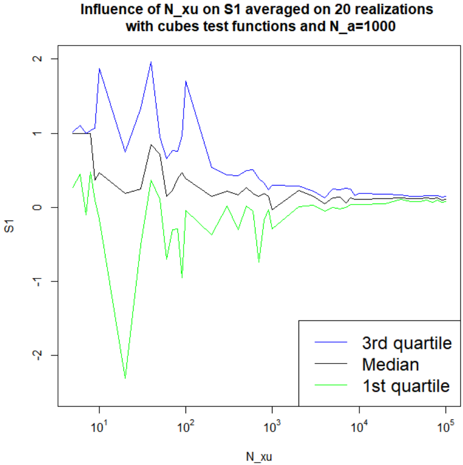
Using $T_a(\Gamma_U) = \mu(\Gamma_U \Delta \gamma_a) = \mathbb{E}_X(\mathbb{1}_{X \in \Gamma \Delta \gamma_a})$ with X uniform on $\mathcal{X}$, we have then

$$\begin{aligned}
\mathrm{Cov}_{U,\tilde{U}(T_a(\Gamma_U), T_a(\Gamma_{\tilde{U}}))} &= \mathbb{E}_{U,\tilde{U}}[\mathbb{E}_X(\mathbb{1}_{X \in \Gamma_U \Delta \gamma_a})\mathbb{E}_X(\mathbb{1}_{X \in \Gamma_{\tilde{U}} \Delta \gamma_a})] \\
&\quad - \mathbb{E}_U[\mathbb{E}_X(\mathbb{1}_{X \in \Gamma_U \Delta \gamma_a})]\mathbb{E}_{\tilde{U}}[\mathbb{E}_X(\mathbb{1}_{X \in \Gamma_{\tilde{U}} \Delta \gamma_a})] \\
&= \mathbb{E}_{X,X',U,\tilde{U}}[\mathbb{1}_{X \in \Gamma_U \Delta \gamma_a} \mathbb{1}_{X' \in \Gamma_{\tilde{U}} \Delta \gamma_a}] \\
&\quad - \mathbb{E}_{X,U}[\mathbb{1}_{X \in \Gamma_U \Delta \gamma_a}]\mathbb{E}_{X',\tilde{U}}[\mathbb{1}_{X' \in \Gamma_{\tilde{U}} \Delta \gamma_a}] \\
&= \frac{1}{N_{xu}} \sum_{j=1}^{N_{xu}} [\mathbb{1}_{X^j \in \Gamma_{U^j} \Delta \gamma_a} \mathbb{1}_{X'^j \in \Gamma_{\tilde{U}^j} \Delta \gamma_a}] \\
&\quad - \frac{1}{N_{xu}} \sum_{j=1}^{N_{xu}} [\mathbb{1}_{X^j \in \Gamma_{U^j} \Delta \gamma_a}] \frac{1}{N_{xu}} \sum_{j=1}^{N_{xu}} [\mathbb{1}_{X'^j \in \Gamma_{\tilde{U}^j} \Delta \gamma_a}].
\end{aligned}$$

Results

Toy function

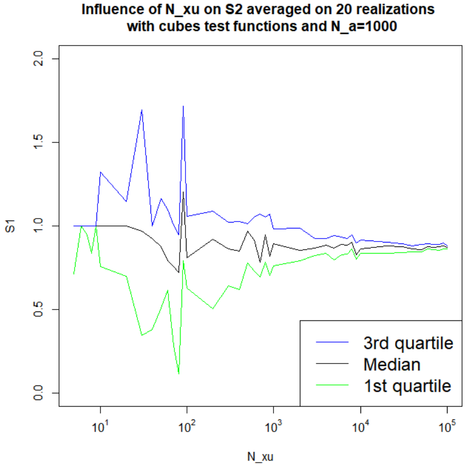
$$\forall(x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$



Results

Toy function

$$\forall (x,u) \in [-5,5]^4 \;\; g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$



Results

With $N_{xu} = 10^6$ and $N_a = 1000$

g function	Index	Results with squares	Results with disks
$-x_1^2 + 5x_2 - u_1 + u_2^2 - 1$	s_1	0.114	0.113
	s_2	0.868	0.864
	$s_1 + s_2$	0.982	0.978
$-x_1^2 + 5x_2 + u_1^2 + u_2^2 - 1$	s_1	0.468	0.450
	s_2	0.461	0.461
	$s_1 + s_2$	0.929	0.911
$-x_1^2 + 5x_2 + u_1^2 + u_2^2 + u_1 u_2 - 1$	s1	0.286	0.263
	s_2	0.283	0.271
	$s_1 + s_2$	0.570	0.535
$-x_1^2 + 5x_2 + u_1^2 - 1$	s_1	1.00	1.00
	s_2	-0.022	0.031
	$s_1 + s_2$	0.977	1.031
$-x_1^2 + 5x_2 + u_1^2 + u_1 u_2 - 1$	s_1	0.426	0.412
	s_2	0.011	-0.000
	$s_1 + s_2$	0.438	0.412
$-x_1^2 + 5x_2 + u_1 + u_2 - 1$	s_1	0.473	0.445
	s_2	0.500	0.438
	$s_1 + s_2$	0.974	0.884

Comments on the results

Toy function

$$\forall (x,u) \in [-5,5]^4 \ g(\boldsymbol{x},\boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

$$\Gamma_U = \{x \in [-5,5]^2, -x_1^2 + 5x_2 \leq u_1 - u_2^2 + 1\}$$

- S^1 and S^2 don't sum to 1 because there are interactions between u_1 and u_2 in Γ_U :
  Indead, two admissible sets are equals if $u_1 - u_2^2 + 1$ is constant which links u_1 and u_2.
- Use of the universal index and the choice of T_a as the symmetric difference gives an index which seems to quantify the influence of the input as observed initially.

Conclusion on the universal index on sets

Choices of T_a and $\mathbb{Q}$?

Choice of T_a

T_a should be chosen so that the $(T_a(\Gamma_U)_{a\in\mathcal{A}})$ characterizes the most Γ_U.

- Choice of T : could be on other distance between sets or a function of a distance.
- Choice of the test sets : We want that the test sets cover the whole space of sets. For instance taking $a \in \mathbb{R}^2$ to defined rectangles could be better than squares.
- The best choice of the test sets could depend on the function g.

Choice of $\mathbb{Q}$

- uniform to cover the whole space of sets.

Conclusion

Vorob'ev index

- Sensitivity indices with sets output : Sobol indices with the Vorob'ev expectation and deviation
- Very costly to estimate and no variance decomposition

$\Rightarrow$ Meta-models adapted to the index

Universal sensitivity indices on sets

- Sensitivity indices with sets outputs : Variance of the transformation of the set output explained by an input averaged on a family of test functions
- Choices of the test function, the probability $\mathbb{Q}$ and their influence on the index

El-Amri, Reda et al. (2021). *A sampling criterion for constrained Bayesian optimization with uncertainties*. arXiv : 2103.05706 [stat.ML].

Fort, Jean-Claude, Thierry Klein et Agnès Lagnoux (jan. 2021). “Global Sensitivity Analysis and Wasserstein Spaces”. In : *SIAM/ASA Journal on Uncertainty Quantification* 9.2. Publisher : Society for Industrial and Applied Mathematics, p. 880-921. doi : 10.1137/20M1354957. url : https://epubs.siam.org/doi/abs/10.1137/20M1354957.

Gamboa, Fabrice et al. (août 2021). “Sensitivity analysis in general metric spaces”. In : *Reliability Engineering and System Safety*. doi : 10.1016/j.ress.2021.107611. url : https://hal.archives-ouvertes.fr/hal-02044223.

Molchanov, Ilya (2005). *Theory of random sets*. T. 19. 2. Springer.

Excursion sets are random sets

For K compact, $h(U) = X = \{x \in \mathcal{X}, g(x, U) \leq 0\}$

$$\begin{aligned}
\{K \cap X \neq \emptyset\} &=^c \{\omega, K \cap X(\omega) = \emptyset\} \\
&=^c \{\omega, \forall x \in K \; g(x, U(\omega)) > 0\} \\
&=^c \{\omega, \inf_{x \in K} g(x, U(\omega)) > 0\} \text{ as } K \text{ compact and g continuous in } x \\
&=^c U^{-1}(\inf_{x \in K} g(x, \cdot)^{-1}(]0, +\infty[)) \in \mathcal{F}
\end{aligned}$$

Universal index on random sets

Expectation of $T_a(\Gamma_U)$

As the randomness of Γ_U only depend on U, we have

$$\mathbb{E}[T_a(\Gamma_U)] = \int_{\mathcal{U}} T_a(\Gamma_U)du. \quad (12)$$

Theorem (Robbins' theorem Molchanov 2005)

Let Γ be a random closed set in a Polish space $\mathcal{X}$. If μ is a locally finite measure on Borel sets, then $\mu(\Gamma)$ is a random variable and :

$$\mathbb{E}(\mu(\Gamma)) = \int_{\mathcal{X}} \mathbb{P}(x \in \Gamma)\mu(dx). \quad (13)$$

$$\begin{aligned}\mathbb{E}(T_a(\Gamma_U)) = \mathbb{E}(\mu(\Gamma_U \Delta \gamma_a)) = \int_{\mathcal{X}} \mathbb{P}(x \in \Gamma_U \Delta \gamma_a)\mu(dx) &= \int_{\mathcal{X}} \mathbb{P}_U(x \in \Gamma_U \Delta \gamma_a)\mu(dx) \\ &= \int_{\mathcal{X}} \int_{\mathcal{U}} \mathbb{1}_{x \in \Gamma_U \Delta \gamma_a} \mu(dx) \\ &= \int_{\mathcal{U}} T_a(\Gamma_u)du.\end{aligned}$$